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Topological sigma models with H-flux

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Abstract: We investigate the topological theory obtained by twisting the $\mathcal{N} = (2, 2)$ supersymmetric nonlinear sigma model with target a bihermitian space with torsion. For the special case in which the two complex structures commute, we show that the action is a Q-exact term plus a quasi-topological term. The quasi-topological term is locally given by a closed two-form which corresponds to a flat gerbe-connection and generalises the usual topological term of the A-model. Exponentiating it gives a Wilson surface, which can be regarded as a generalization of a Wilson line. This makes the quantum theory globally well-defined.

Keywords: Extended Supersymmetry, Topological Strings, Sigma Models, Topological Field Theories.

Contents

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1. Introduction

The topological sigma model was introduced in [1] for a target which is a symplectic manifold. In [2, 3] it was observed that for the Kähler case the topological sigma model can be obtained by twisting the $\mathcal{N} = (2, 2)$ sigma model with Kähler target, with a certain linear combination of the four supercharges becoming a scalar charge Q, sometimes referred to as a BRST operator. The action is the sum of a Q-exact term and a topological term, so that the path integral is given as a sum over topological sectors weighted by the exponential of the topological term. Moreover the path integral of the twisted model is localized on the fixed points of the Q-action. For the A-model of [2], the theory localises on holomorphic maps and the topological term is the pullback of the Kähler form. For a comprehensive review and applications to string theory see, e.g., [4].

The most general $\mathcal{N} = (2, 2)$ sigma model has both a kinetic term given by a metric g and a Wess-Zumino term defined by a closed 3-form H, so that in each patch U_α there is a 2-form B_α with $H = dB_\alpha$ [5, 6]. The geometry of the target space is bihermitian, i.e., the metric g is hermitian with respect to two complex structures $J_\pm$, and these are covariantly constant with respect to the covariant derivatives with torsion

$$\nabla^{(\pm)} J_\pm = 0\,, \qquad \nabla^{(\pm)} = \nabla \pm \frac{1}{2}\, g^{-1} H\,, \tag{1.1}$$

where ∇ is the Levi-Civita connection. Alternatively, the conditions (1.1) can be rewritten as the following integrability conditions

$$H = d_+^c \omega_+ = -d_-^c \omega_-\,, \tag{1.2}$$

where $\omega_\pm = gJ_\pm$ and $d_\pm^c$ are the $i(\bar\partial - \partial)$ operators for the complex structures $J_\pm$. When the complex structures commute, the geometry also carries a local product structure. We refer to this special case as a bihermitian Local Product (BiLP) geometry. It was shown in [5] that this case has a manifest $\mathcal{N} = (2, 2)$ superspace description in terms of chiral and twisted chiral superfields. Bihermitian geometry has been given an alternative formulation as Generalized Kähler Geometry [7] and subsequently in [8] it has been demonstrated that this geometry is locally described in $\mathcal{N} = (2, 2)$ superspace using semi-(anti) chiral superfields [9] in addition to chiral and twisted ones.

The only term explicitly dependent on B is the the WZ-term, which is proportional to

$$\int X^*(B) = \frac{1}{2} \int d^2\xi \; B_{AB}\epsilon^{\mu\nu}\partial_\mu X^A \partial_\nu X^B \; .$$

The terms involving fermions depend on B only through the field strength H. In Euclidean signature, Wick-rotation leads to an imaginary WZ term with a factor of 'i' in front of B.

For the quantum theory to be well-defined, it is necessary that $H \in H^3(M, \mathbb{Z})$, so that there is a gerbe with connection $\{B_\alpha\}$ whose curvature is H. For the path integral, if $H_2(M)$ is trivial, then the image $c_2 = X(\Sigma_2)$ of a compact world-sheet (Σ_2 is the boundary of a three dimensional submanifold c_3) and the WZ term can be written as an integral of H over c_3, so that the WZ-term depends on H only. If $H_2(M)$ is non-trivial, it is not sufficient to specify H, and a choice of B must be specified. Then the exponent of the WZ-term is defined as the holonomy of a gerbe over $X(\Sigma_2)$, see, e.g., formula in the appendix (for further details on gerbes and gerbe holonomy the reader may consult [10 − 13]). It is important to stress that to define a gerbe holonomy we in general need full information about the gerbe connection $\{B_\alpha\}$, not just H alone.

The twisting of the general $\mathcal{N} = (2, 2)$ sigma model with torsion was considered in [14] and discussed further in [15 − 17]. However, there were problems in writing the action as the sum of a Q-exact term and a topological term, so that it was hard to understand the structure of the path integral as a weighted sum. Here we shall write the action in just such a form, in the special case in which the complex structures commute, so that the geometry is BiLP. The Q-exact term QV can be found from the $\mathcal{N} = (2, 2)$ superspace formulation, in which the action is given by the superspace integral of a potential K depending on all superfields. Grassmann coordinates $\theta, \theta^1, \theta^2, \theta^3$ can be chosen such that $Q = \partial/\partial\theta$, and hence V is given by integrating K over $\theta^1, \theta^2, \theta^3$. Strictly speaking, it is determined in this way up to a total derivative term, as the usual superspace approach is not sufficiently careful with boundary terms. Although we believe that twisting can nevertheless be performed in superspace [18], we choose to circumvent the derivative question and related issues by using a component presentation in which the total derivative terms are fixed.

The local product structure splits the coordinates locally into two sets, ϕ^i and χ^a. One set of coordinates are the leading components of chiral superfields and the other set are the leading components of twisted chiral superfields; which can be interchanged by a coordinate redefinition in superspace. The A-twist of the model in which the ϕ^i are chiral and the χ^a are twisted chiral is the same as the B-twist of the model in which the ϕ^i are twisted chiral and the χ^a are chiral, so all cases are covered by considering, say, the B-twist of the general model with arbitrary numbers of chiral superfields and twisted chiral

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superfields.[1] We shall discuss the B-twist here.

The main result of the paper can be summarized as follows. The action of the twisted model can be written as a sum of Q-exact term and a 'quasi-topological term'. This reduces to the usual topological term of the topological sigma-model when the target space is Kahler, but with a non-trivial B field, this term in the action is not well-defined. However, its exponential is well-defined and gives the holonomy of a flat gerbe, so that the quantum theory is well-defined, with the path integral weighted by these holonomies.

The structure of the paper is as follows. In section 2 we present some background information about the superspace formalism and topological twist. In section 3 the component analysis is done. The twisted action is written as a sum of a Q-exact term and the pull-back of a locally defined closed form. Section 4 is devoted to the geometrical interpretation of the quasi-topological term using the language of flat gerbes. In section 5 some comments and speculations are presented. In appendix we briefly review the definition of holonomy for line bundles and gerbes.

2. Background

In this section we define the twist from the point of view of superspace.

The original $\mathcal{N} = (2, 2)$ sigma model with Minkowski signature has the Lorentz group $\mathrm{SO}(1, 1)$ acting on the world sheet coordinates. In addition, there is an $\mathrm{SO}(2) \times \mathrm{SO}(2)$ R-symmetry acting on the superspace Grassman variables (θ^{1+},θ^{2+},θ^{1-},θ^{2-}), with one $\mathrm{SO}(2)$ acting on the positive chirality odd variables θ^{I+} and the other acting on the negative chirality ones θ^{I-} ($I = 1, 2$). It is important to remember that θ^{I+} and θ^{I-} transform as Majorana-Weyl spinors, i.e. each is a one-component real spinor. The symmetry group of the sigma model with Minkowski signature is then

$$\mathrm{SO}(1, 1) \times \mathrm{SO}(2) \times \mathrm{SO}(2)\ . \tag{2.1}$$

The R-rotations act on superfields Φ as vector or axial rotations:

$$\begin{aligned} V\,:\,\Phi^i(x, \theta^\pm, \bar\theta^\pm) &\mapsto e^{i\alpha q_V^i}\,\Phi^i(x, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar\theta^\pm) \\ A\,:\,\Phi^i(x, \theta^\pm, \bar\theta^\pm) &\mapsto e^{i\beta q_A^i}\,\Phi^i(x, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar\theta^\pm)\,, \end{aligned} \tag{2.2}$$

where q_V and q_A are the vector and axial R-charges respectively. Here $\theta^\pm = \frac{1}{\sqrt{2}}(\theta^{1\pm} + i\theta^{2\pm})$. To twist the model as in [2, 3], one must first Wick rotate, so that the Lorentz group becomes $\mathrm{SO}(2)$.

In Euclidean signature, we want to treat fields and their complex conjugates as formally independent, in order to allow e.g. a B-twist in which positive and negative chirality fields are twisted differently. This means that we need to consider the *complexification* of the Euclidean theory (in analogy with CFT) and as a result consider the complexification of the $\mathcal{N} = (2, 2)$ superspace. We will say more about this complexification elsewhere [19],

[1] Note that T-duality and mirror symmetry interchanges A and B-twists, and thus chiral and twisted chiral fields.

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but here simply note that the complexified $\mathcal{N} = (2,2)$ Euclidean sigma model has the symmetry group (Lorentz and R symmetries)

$$\mathrm{SO}(2,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C})\,, \tag{2.3}$$

allowing the possibility of A and B twists. In the complexified superspace we treat the (twisted) chiral and (twisted) anti-chiral as independent fields.

The $\mathcal{N} = (2,2)$ Euclidean supersymmetry algebra[2] is

$$\begin{aligned} \left\{\mathbb{D}_+, \bar{\mathbb{D}}_+\right\} &= i\partial\,, \\ \left\{\mathbb{D}_-, \bar{\mathbb{D}}_-\right\} &= i\bar{\partial}\,, \end{aligned} \tag{2.4}$$

where $\partial \equiv \partial/\partial z$ and $z = x^1 + ix^2$. As a result of twisting, two of the supercharges become scalars Q, $\hat{D}$, and two become vectors D_z, $D_{\bar{z}}$. For the A and B twists,

$$\begin{aligned} &A - twist: \quad D_z = \bar{\mathbb{D}}_-,\ \ D_{\bar{z}} = \mathbb{D}_+,\ \ Q = \frac{1}{2}\left(\bar{\mathbb{D}}_+ + \mathbb{D}_-\right),\ \ \hat{D} = \frac{1}{2i}\left(\bar{\mathbb{D}}_+ - \mathbb{D}_-\right)\,, \\ &B - twist: \quad D_z = \mathbb{D}_-,\ \ D_{\bar{z}} = \mathbb{D}_+,\ \ Q = \frac{1}{2}\left(\bar{\mathbb{D}}_+ + \bar{\mathbb{D}}_-\right),\ \ \hat{D} = \frac{1}{2i}\left(\bar{\mathbb{D}}_+ - \bar{\mathbb{D}}_-\right)\,. \end{aligned} \tag{2.5}$$

The new operators obey the algebra

$$\{Q, D_z\} = \frac{i}{2}\bar{\partial},\ \ \{Q, D_{\bar{z}}\} = \frac{i}{2}\partial,\ \ \{\hat{D}, D_z\} = -\frac{1}{2}\bar{\partial},\ \ \{\hat{D}, D_{\bar{z}}\} = \frac{1}{2}\partial\,, \tag{2.6}$$

for both twists. In fact, from the superspace point of view only one twist exists, the A and B twists being related by a coordinate transformation exchanging θ^- with $\bar{\theta}^-$, or equivalently, by exchanging the chiral and twisted chiral fields. For concreteness, from now on we shall focus on the B-twist.

In Minkowski superspace, chiral (Φ) and twisted chiral fields (χ) obey the constraints

$$\begin{aligned} \bar{\mathbb{D}}_\pm \Phi = 0\ ,\ \bar{\mathbb{D}}_+\chi = \mathbb{D}_-\chi = 0\,, \\ \mathbb{D}_\pm \bar{\Phi} = 0\ ,\ \mathbb{D}_+\bar{\chi} = \bar{\mathbb{D}}_-\bar{\chi} = 0\,, \end{aligned} \tag{2.7}$$

along with their conjugates. In the Euclidean theory we need to consider ϕ, $\bar{\phi}$, χ, $\bar{\chi}$ as independent with constraints (2.7).

For a B-twist, the chiral constraints on the superfields Φ, $\bar{\Phi}$ may be re-expressed in terms of the new operators as

$$Q\Phi = 0,\ \ \hat{D}\Phi = 0,\ \ D_z\bar{\Phi} = 0,\ \ D_{\bar{z}}\bar{\Phi} = 0\,, \tag{2.8}$$

while the twisted chiral superfields χ, $\bar{\chi}$ obey

$$D_z\chi = 0,\ \ Q\chi = -i\hat{D}\chi,\ \ D_{\bar{z}}\bar{\chi} = 0,\ \ Q\bar{\chi} = i\hat{D}\bar{\chi}\,. \tag{2.9}$$

[2]In the algebra (2.4) it would be natural to remove the imaginary 'i' from the right hand side. However we prefer to keep it in order to preserve the formal similarities with Minkowski-signature superspace.

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Denoting the θ-independent part by a vertical bar, we define the standard components of our superfields as

$$
\begin{aligned}
&\Phi : \phi \equiv \Phi|, \quad \psi_\pm \equiv \mathbb{D}_\pm \Phi|, \quad F \equiv \mathbb{D}_+ \mathbb{D}_- \Phi| \,, \\
&\chi : \chi \equiv \chi|, \quad \lambda_+ \equiv \mathbb{D}_+ \chi|, \quad \lambda_- \equiv \bar{\mathbb{D}}_- \chi|, \quad G \equiv \mathbb{D}_+ \bar{\mathbb{D}}_- \chi| \,,
\end{aligned}
\tag{2.10}
$$

along with similar expressions for $\bar\Phi$ and $\bar\chi$. For completeness we also define the components with respect to the new operators (2.5) and give their relation to the components (2.10)

$$
\begin{aligned}
&\chi : \tilde\chi = \chi| = \chi, \;\; \rho_{\bar z} = D_{\bar z}\chi| = \lambda_+, \;\; \eta = -i\hat D \chi|, = \frac{1}{2}\lambda_- \;\; G_{\bar z} = i D_{\bar z}\hat D \chi| = -\frac{1}{2}G \\
&\bar\chi : \bar{\tilde\chi} = \bar\chi|, = \bar\chi \;\; \bar\rho_z = D_z \bar\chi|, = \bar\lambda_- \;\; \bar\eta = i\hat D\bar\chi| = \frac{1}{2}\bar\lambda_+, \;\; \bar G_z = -iD_z \hat D \bar\chi| = -\frac{1}{2}\bar G, \\
&\Phi : \tilde\phi = \Phi| = \phi, \;\; \psi_z = D_z\Phi| = \psi_-, \;\; \psi_{\bar z} = D_{\bar z}\Phi| = \psi_+, \;\; F_{z\bar z} = D_z D_{\bar z}\Phi| = F, \\
&\bar\Phi : \bar{\tilde\phi} = \bar\Phi| = \bar\phi, \;\; \bar\varphi = Q\bar\Phi| = \frac{1}{2}(\bar\psi_+ + \bar\psi_-), \;\; \bar\zeta = \hat D\bar\Phi| = -\frac{i}{2}(\bar\psi_+ - \bar\psi_-) \,, \\
&\qquad \bar H = Q\hat D\bar\Phi| = -\frac{i}{2}\bar F \;.
\end{aligned}
\tag{2.11}
$$

We choose to express the $BRST$ transformations (generated by the charge Q) in terms of the non-standard components:

$$
\begin{aligned}
&\chi : \delta\chi = \eta, \;\; \delta\eta = 0, \;\; \delta\rho_{\bar z} = G_{\bar z} + \frac{i}{2}\partial\chi, \;\; \delta G_{\bar z} = -\frac{i}{2}\partial\eta \,, \\
&\bar\chi : \delta\bar\chi = \bar\eta, \;\; \delta\bar\eta = 0, \;\; \delta\bar\rho_z = \bar G_z + \frac{i}{2}\bar\partial\bar\chi, \;\; \delta\bar G_z = -\frac{i}{2}\bar\partial\bar\eta, \\
&\Phi : \delta\tilde\phi = 0, \;\; \delta\psi_z = \frac{i}{2}\bar\partial\tilde\phi, \;\; \delta\psi_{\bar z} = \frac{i}{2}\partial\tilde\phi, \;\; \delta F_{z\bar z} = -\frac{i}{2}(\partial\psi_z - \bar\partial\psi_{\bar z}), \\
&\bar\Phi : \delta\bar{\tilde\phi} = \bar\varphi, \;\; \delta\bar\varphi = 0, \;\; \delta\bar\zeta = \bar H, \;\; \delta\bar H = 0.
\end{aligned}
\tag{2.12}
$$

We now have the option of continuing our analysis in superspace where the Q exact term in the Lagrangian can be derived once we have decided which set of components to use. This line of attack will be followed elsewhere [18]. Alternatively we may turn directly to a component treatment which is what we do in the next section. If we do not use complex conjugation in our calculations then the formal manipulations in Minkowski and complexified Euclidean space or superspace are exactly the same.

3. The twisted model

The approach taken in this section is to start from a specific form of the component Lagrangian and then carefully keep track of all total derivative contributions under variations. The component approach also allows us to keep the considerations general enough to allow, e.g., for almost complex geometries.

The supersymmetry transformations we need follow from the superspace transformation rule

$$
\delta\Sigma = [\alpha_- Q_+ + \alpha_+ Q_- + \tilde\alpha_- \bar Q_+ + \tilde\alpha_+ \bar Q_-, \Sigma] \,,
\tag{3.1}
$$

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for any superfield Σ. Since we will be interested in the transformations of the components arrived at by taking θ-independent parts of various superfields, we may replace the supercharges in (3.1) by covariant derivatives according to:

$$\delta\Sigma| = i[\alpha_-\mathbb{D}_+ + \alpha_+\mathbb{D}_- + \tilde{\alpha}_-\bar{\mathbb{D}}_+ + \tilde{\alpha}_+\bar{\mathbb{D}}_-, \Sigma]|\,, \tag{3.2}$$

where the vertical bar denotes the θ-independent part. Using (3.2) for the superfield Σ given by Φ, $D\Phi$, $D^2\Phi$, gives the transformations for the chiral multiplet used in [2],

$$\begin{aligned}
\delta\phi^i &= i\alpha_-\psi_+^i + i\alpha_+\psi_-^i\,,\\
\delta\phi^{\bar{i}} &= i\tilde{\alpha}_-\psi_+^{\bar{i}} + i\tilde{\alpha}_+\psi_-^{\bar{i}}\,,\\
\delta\psi_+^i &= -\tilde{\alpha}_-\partial\phi^i - i\alpha_+F^i\,,\\
\delta\psi_+^{\bar{i}} &= -\alpha_-\partial\phi^{\bar{i}} + i\tilde{\alpha}_+F^{\bar{i}}\,,\\
\delta\psi_-^i &= -\tilde{\alpha}_+\bar{\partial}\phi^i + i\alpha_-F^i\,,\\
\delta\psi_-^{\bar{i}} &= -\alpha_+\bar{\partial}\phi^{\bar{i}} - i\tilde{\alpha}_-F^{\bar{i}}\,,\\
\delta F^i &= -\tilde{\alpha}_-\partial\psi_-^i + \tilde{\alpha}_+\bar{\partial}\psi_+^i\,,\\
\delta F^{\bar{i}} &= \alpha_-\partial\psi_-^{\bar{i}} - \alpha_+\bar{\partial}\psi_+^{\bar{i}}\,,
\end{aligned} \tag{3.3}$$

where we have introduced a set of chiral multiplets labeled by $i, j, \ldots$ and a set of antichiral multiplets labelled by $\bar{i}$, $\bar{j}$, ... The transformations of the components of twisted multiplets labelled by $a, b, \ldots$ and anti twisted multiplets labelled by $\bar{a}$, $\bar{b}$, ... are found similarly to be

$$\begin{aligned}
\delta\chi^a &= i\alpha_-\lambda_+^a + i\tilde{\alpha}_+\lambda_-^a\,,\\
\delta\chi^{\bar{a}} &= i\alpha_+\lambda_-^{\bar{a}} + i\tilde{\alpha}_-\lambda_+^{\bar{a}}\,,\\
\delta\lambda_+^a &= -\tilde{\alpha}_-\partial\chi^a - i\tilde{\alpha}_+G^a\,,\\
\delta\lambda_+^{\bar{a}} &= -\alpha_-\partial\chi^{\bar{a}} + i\alpha_+G^{\bar{a}}\,,\\
\delta\lambda_-^a &= -\alpha_+\bar{\partial}\chi^a + i\alpha_-G^a\,,\\
\delta\lambda_-^{\bar{a}} &= -\tilde{\alpha}_+\bar{\partial}\chi^{\bar{a}} - i\tilde{\alpha}_-G^{\bar{a}}\,,\\
\delta G^a &= -\tilde{\alpha}_-\partial\lambda_-^a + \alpha_+\bar{\partial}\lambda_+^a\,,\\
\delta G^{\bar{a}} &= \alpha_-\partial\lambda_-^{\bar{a}} - \tilde{\alpha}_+\bar{\partial}\lambda_+^{\bar{a}}\,.
\end{aligned} \tag{3.4}$$

As shown in [5], the $\mathcal{N} = (2,2)$ Lagrangian used in [2] can be generalized to include the twisted chiral fields. It then reads

$$S = \int d^2\xi\left(E_{AB}\partial X^A\bar{\partial}X^B + \frac{1}{2}g_{AB}\psi_+^A i\nabla^{(+)}\psi_+^B + \frac{1}{2}g_{AB}\psi_-^A i\bar{\nabla}^{(-)}\psi_-^B + \frac{1}{4}R^{(+)}_{ABCD}\psi_+^A\psi_+^B\psi_-^C\psi_-^D\right), \tag{3.5}$$

where $A, B, \ldots$ label the (anti)chiral and twisted (anti)chiral fields and the connection used in the covariant derivative and curvature have torsion $\Gamma^{(\pm)}_{ABC} \equiv \Gamma^{(0)}_{ABC} \pm \frac{1}{2}H_{ABC}$ with $\Gamma^{(0)}$ the Levi-Civita connection and

$$H \equiv dB\,, \quad E_{AB} \equiv g_{AB} + B_{AB}\,. \tag{3.6}$$

The explicit form of the action may be found using the components defined in (2.10).

The equations of motion for the auxiliary fields F and G are,

$$\begin{aligned} K_{\bar{j}i}F^i &= K_{\bar{j}kl}\psi^l_-\psi^k_+ + K_{\bar{j}a\bar{b}}\lambda^{\bar{b}}_-\lambda^a_+ + K_{\bar{j}k\bar{a}}\lambda^{\bar{a}}_-\psi^k_+ + K_{\bar{j}ak}\psi^k_-\lambda^a_+ \\ K_{\bar{b}a}G^a &= K_{\bar{b}cd}\lambda^d_-\lambda^c_+ + K_{\bar{b}i\bar{j}}\psi^{\bar{j}}_-\psi^i_+ + K_{\bar{b}c\bar{i}}\psi^{\bar{i}}_-\lambda^c_+ + K_{\bar{b}ic}\lambda^c_-\psi^i_+, \end{aligned} \tag{3.7}$$

and may be used to go partially on-shell and eliminate the auxiliary fields in (3.3) and (3.4).

To construct a topological model out of this Lagrangian, we use the same procedure as in [2], and twist the Lorentz group with either a vector or axial subgroup of the R-symmetry group, so that two of the supercharges become scalars.

Using the R-symmetries available in the (2,2)-algebra we twist the Lorentz transformations of the fields, such that for the B-model

$$\begin{aligned} Q_B &= \bar{Q}_+ + \bar{Q}_- \,, \\ Q^T &= \bar{Q}_+ - \bar{Q}_- \,, \end{aligned} \tag{3.8}$$

are scalar charges.

As in [2], we expect to obtain a topological model after the twisting with a Lagrangian L_{total} written

$$L_{\text{total}} = L_B + L_{\text{top}} \ . \tag{3.9}$$

The first term L_B would be a Q_B exact term ($L_B = Q_B V'$). The second term L_{top} is expected to be some kind of 'topological term' that is a total derivative and does not affect the equations of motion. If this term is given by a closed 2-form, then its integral is a topological invariant depending on the cohomology class of the 2-form and the homology class of the embedding of the world-sheet in the target. In the case involving only chiral and antichiral fields, for the A-model, this term turns out to be the Kähler form ω of the target space and its integral is the degree of the holomorphic map, [2]. To find the total derivative term here, we will first calculate the exact part L_B of the Lagrangian and subtract it from the full Lagrangian L_{total}. Here we find it not to be given by a globally defined 2-form, but instead by a flat gerbe connection, so it does not determine a cohomology class but instead leads to a generalisation of a Wilson line.

Then V' is calculated by taking a potential depending on all the fields, $K(\phi, \bar{\phi}, \chi, \bar{\chi})$, and acting on it with the other 3 supersymmetries,

$$V' = Q^T Q_- Q_+ K \ . \tag{3.10}$$

(We note in passing that L_B is proportional to the full superspace integral of the superspace Lagrangian $K(\Phi, \bar{\Phi}, \chi, \bar{\chi})$ up to surface terms which depend on the precise prescription for the superspace measure.) We can use the transformations (3.4) and (3.5) to perform the first two steps, and use the transformations for the scalar supercharges,

$$
\begin{aligned}
\delta\phi^{\bar{i}} &= i\alpha_B(\psi_+^{\bar{i}} + \psi_-^{\bar{i}}) + i\alpha^T(\psi_-^{\bar{i}} - \psi_+^{\bar{i}})\,, \\
\delta\psi_+^i &= -\alpha_B\partial\phi^i + \alpha^T\partial\phi^i\,, \\
\delta\psi_-^i &= -\alpha_B\bar{\partial}\phi^i - \alpha^T\bar{\partial}\phi^i\,, \\
\delta F^i &= -\alpha_B(\partial\psi_-^i - \bar{\partial}\psi_+^i) + \alpha^T(\partial\psi_-^i + \bar{\partial}\psi_+^i) \\
\delta\chi^a &= i\alpha_B\lambda_-^a + i\alpha^T\lambda_-^a\,, \\
\delta\chi^{\bar{a}} &= i\alpha_B\lambda_+^{\bar{a}} - i\alpha^T\lambda_+^{\bar{a}}\,, \\
\delta\lambda_+^a &= \alpha_B(-\partial\chi^a - iG^a) + \alpha^T(\partial\chi^a - iG^a)\,, \\
\delta\lambda_-^{\bar{a}} &= \alpha_B(-\bar{\partial}\chi^{\bar{a}} - iG^{\bar{a}}) + \alpha^T(-\bar{\partial}\chi^{\bar{a}} + iG^{\bar{a}})\,,
\end{aligned}
\tag{3.11}
$$

for the last variation of the potential K.

Performing these variations, we find the explicit form of V',

$$
\begin{aligned}
V' = & -iK_{i\bar{j}}\psi_-^{\bar{j}}F^i + iK_{i\bar{j}}\psi_+^{\bar{j}}F^i - iK_{ia}\lambda_-^aF^i + iK_{i\bar{a}}\lambda_+^{\bar{a}}F^i \\
& -K_i\partial\psi_-^i - K_i\bar{\partial}\psi_+^i - iK_{ij\bar{k}}\psi_-^{\bar{k}}\psi_+^j\psi_-^i + iK_{ij\bar{k}}\psi_+^{\bar{k}}\psi_+^j\psi_-^i \\
& -iK_{ija}\lambda_-^a\psi_+^j\psi_-^i + iK_{ij\bar{a}}\lambda_+^{\bar{a}}\psi_+^j\psi_-^i - K_{ij}\partial\phi^j\psi_-^i - K_{ij}\psi_+^j\bar{\partial}\phi^i \\
& -iK_{\bar{a}b\bar{i}}\psi_-^{\bar{i}}\lambda_+^b\lambda_-^{\bar{a}} + iK_{\bar{a}b\bar{i}}\psi_+^{\bar{i}}\lambda_+^b\lambda_-^{\bar{a}} - iK_{\bar{a}bc}\lambda_-^c\lambda_+^b\lambda_-^{\bar{a}} + iK_{\bar{a}b\bar{c}}\lambda_+^{\bar{c}}\lambda_+^b\lambda_-^{\bar{a}} \\
& -K_{\bar{a}b}\partial\chi^b\lambda_-^{\bar{a}} + iK_{\bar{a}b}G^b\lambda_-^{\bar{a}} - K_{\bar{a}b}\lambda_+^b\bar{\partial}\chi^{\bar{a}} + iK_{\bar{a}b}\lambda_+^bG^{\bar{a}} \\
& -iK_{ia\bar{j}}\psi_-^{\bar{j}}\lambda_+^a\psi_-^i + iK_{ia\bar{j}}\psi_+^{\bar{j}}\lambda_+^a\psi_-^i - iK_{iab}\lambda_-^b\lambda_+^a\psi_-^i + iK_{ia\bar{b}}\lambda_+^{\bar{b}}\lambda_+^a\psi_-^i \\
& -K_{ia}\partial\chi^a\psi_-^i + iK_{ia}G^a\psi_-^i - K_{ia}\lambda_+^a\bar{\partial}\phi^i \\
& -iK_{\bar{a}i\bar{j}}\psi_-^{\bar{j}}\psi_+^i\lambda_-^{\bar{a}} + iK_{\bar{a}i\bar{j}}\psi_+^{\bar{j}}\psi_+^i\lambda_-^{\bar{a}} - iK_{\bar{a}ib}\lambda_-^b\psi_+^i\lambda_-^{\bar{a}} + iK_{\bar{a}i\bar{b}}\lambda_+^{\bar{b}}\psi_+^i\lambda_-^{\bar{a}} \\
& -K_{\bar{a}i}\partial\phi^i\lambda_-^{\bar{a}} - K_{\bar{a}i}\psi_+^i\bar{\partial}\chi^{\bar{a}} + iK_{\bar{a}i}\psi_+^iG^{\bar{a}}\ .
\end{aligned}
\tag{3.12}
$$

Here indices on K denote partial derivatives, so that e.g.

$$
K_{i\bar{j}} = \partial_i\partial_{\bar{j}}K\ .
$$

To calculate the exact term, we still have to $BRST$-transform V'. Just as in the chiral case, the topological term will be associated with the part of the action involving only the scalars. For Q_BV' the purely bosonic part is given by

$$
\begin{aligned}
Q_BV' = & K_{i\bar{j}}\partial\phi^{\bar{j}}\bar{\partial}\phi^i + K_{i\bar{j}}\bar{\partial}\phi^{\bar{j}}\partial\phi^i \\
& -K_{\bar{a}b}\partial\chi^b\bar{\partial}\chi^{\bar{a}} - K_{\bar{a}b}\partial\chi^b\bar{\partial}\chi^{\bar{a}} \\
& +K_{i\bar{a}}\partial\chi^{\bar{a}}\bar{\partial}\phi^i + K_{ia}\bar{\partial}\chi^a\partial\phi^i \\
& -K_{ia}\partial\chi^a\bar{\partial}\phi^i - K_{\bar{a}i}\partial\phi^i\bar{\partial}\chi^{\bar{a}} + \text{fermi terms}\ .
\end{aligned}
\tag{3.13}
$$

Note that the fixed points of the Q-transformations are given by holomorphic twisted chiral maps and by constant chiral ones. Thus arguments show that the theory is localized on those maps.

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Let us now focus on the full Lagrangian of the system. We know from [5] that the target space is a bihermitian manifold. We choose a local chart for that manifold such that

$$\begin{aligned}
J_{\pm}{}^{i}{}_{j} &= i\delta^{i}{}_{j}\,, \\
J_{\pm}{}^{\bar{i}}{}_{\bar{j}} &= -i\delta^{\bar{i}}{}_{\bar{j}}\,, \\
J_{+}{}^{a}{}_{b} &= i\delta^{a}{}_{b}\,, \qquad J_{-}{}^{a}{}_{b} = -i\delta^{a}{}_{b}\,, \\
J_{+}{}^{\bar{a}}{}_{\bar{b}} &= -i\delta^{\bar{a}}{}_{\bar{b}}\,, \quad\; J_{-}{}^{\bar{a}}{}_{\bar{b}} = i\delta^{\bar{a}}{}_{\bar{b}}\,,
\end{aligned} \tag{3.14}$$

where $J_{\pm}$ are the complex structures of the manifold. The metric g is

$$g_{i\bar{j}} = K_{i\bar{j}} \quad g_{a\bar{b}} = -K_{a\bar{b}}\ . \tag{3.15}$$

We then define the two-forms $\omega_{\pm} = gJ_{\pm}$, which in this coordinate system are

$$\omega_{\pm} = -iK_{i\bar{j}}d\phi^{i}\wedge d\phi^{\bar{j}} \pm iK_{a\bar{b}}d\chi^{a}\wedge d\chi^{\bar{b}}\ . \tag{3.16}$$

The full Lagrangian L_{total} has two distinct geometrical parts, terms depending on the metric g of the target space and terms depending on a B-field on the target space. For the B-field, a useful gauge is the one of [5]. In this gauge, B is chosen to be B_{-}, where $dB_{-} = H$ and the $(1,1)$ component of B_{-} with respect to J_{-} vanishes. Then with $B = B_{-}$, the bosonic part of L_{total} reads

$$L_{\text{total}} = K_{i\bar{j}}\partial\phi^{i}\bar{\partial}\phi^{\bar{j}} + K_{i\bar{j}}\bar{\partial}\phi^{i}\partial\phi^{\bar{j}} - K_{a\bar{b}}\partial\chi^{a}\bar{\partial}\chi^{\bar{b}} - K_{a\bar{b}}\bar{\partial}\chi^{a}\partial\chi^{\bar{b}} - K_{i\bar{a}}d\phi^{i}\wedge d\chi^{\bar{a}} - K_{\bar{i}a}d\phi^{\bar{i}}\wedge d\chi^{a}\ . \tag{3.17}$$

We can write L_{total} explicitly as

$$\begin{aligned}
L_{\text{total}} = &\ K_{i\bar{j}}\partial\phi^{i}\bar{\partial}\phi^{\bar{j}} + K_{i\bar{j}}\bar{\partial}\phi^{i}\partial\phi^{\bar{j}} \\
&-K_{a\bar{b}}\partial\chi^{a}\bar{\partial}\chi^{\bar{b}} - K_{a\bar{b}}\bar{\partial}\chi^{a}\partial\chi^{\bar{b}} \\
&-K_{i\bar{a}}\partial\phi^{i}\bar{\partial}\chi^{\bar{a}} - K_{\bar{i}a}\partial\phi^{\bar{i}}\bar{\partial}\chi^{a} \\
&+K_{a\bar{i}}\partial\chi^{a}\bar{\partial}\phi^{\bar{i}} + K_{\bar{a}i}\partial\chi^{\bar{a}}\bar{\partial}\phi^{i}\ .
\end{aligned} \tag{3.18}$$

We are now ready to determine the total derivative term of the action L_{top}. Subtracting the exact part of the action $Q_{B}V'$, given in (3.13), from L_{total}, given in (3.18), we get

$$\begin{aligned}
L_{\text{top}} = &\ K_{a\bar{b}}\partial\chi^{a}\bar{\partial}\chi^{\bar{b}} - K_{a\bar{b}}\bar{\partial}\chi^{a}\partial\chi^{\bar{b}} \\
&+K_{ai}\partial\chi^{a}\bar{\partial}\phi^{i} - K_{ai}\bar{\partial}\chi^{a}\partial\phi^{i} \\
&+K_{a\bar{i}}\partial\chi^{a}\bar{\partial}\phi^{\bar{i}} - K_{a\bar{i}}\bar{\partial}\chi^{a}\partial\phi^{\bar{i}}\ .
\end{aligned} \tag{3.19}$$

Focusing on the target space, the term can be written as a pullback of a two-form,

$$L_{\text{top}} = X^{*}(K_{a\bar{b}}d\chi^{a}\wedge d\chi^{\bar{b}} - K_{ia}d\phi^{i}\wedge d\chi^{a} - K_{\bar{i}a}d\phi^{\bar{i}}\wedge d\chi^{a})\,, \tag{3.20}$$

and we can see that the term is locally exact

$$L_{\text{top}} = -X^{*}\left(d(K_{a}d\chi^{a})\right)\ . \tag{3.21}$$

In other words, this is a total derivative term, but is not globally defined.

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4. Gerbes and the total derivative term

In this section we use gerbes (see appendix for some basic facts) to elaborate on the geometrical meaning of the quasi-topological term[3] (3.21). We argue that although the action is not well-defined, the path integral is.

Before proceeding, it will be useful to compare two gauge choices for the B-field. In a patch U_α, we can choose the gauge $(B_\alpha) = (B_\alpha)_+$ in which the (1,1) part of B with respect to J_+ vanishes, or the gauge $(B_\alpha) = (B_\alpha)_-$ used in [5] in which the (1,1) part of B with respect to J_- vanishes, so that the gauges for $B_\pm$ are

$$(B_\alpha)_\pm = (B_\alpha)_\pm^{(2,0)} + (B_\alpha)_\pm^{(0,2)} , \tag{4.1}$$

with $B_\pm^{(1,1)}$ is zero. (Explicitly, $(B_\alpha)_+^{(2,0)}$ is the (2,0) part of B with respect to J_+ and $(B_\alpha)_-^{(2,0)}$ is the (2,0) part of B with respect to J_-.) These two gauge choices differ by a globally defined exact form

$$(B_\alpha)_+ - (B_\alpha)_- = 2d\Lambda_\alpha \tag{4.2}$$

where $\Lambda_\alpha = \Lambda$ is a global 1-form.

As H is of type $(2,1) + (1,2)$ with respect to both complex structures, it follows that

$$H_\pm^{(2,1)} = d(B_\alpha)_\pm^{(2,0)} \ . \tag{4.3}$$

In the coordinate system used in previous sections, the explicit form of $(B_\alpha)_\pm^{(2,0)}$ is

$$(B_\alpha)_-^{(2,0)} = K_{\bar{b}i}\ d\chi^{\bar{b}} \wedge d\phi^i , \qquad (B_\alpha)_+^{(2,0)} = K_{ia}\ d\phi^i \wedge d\chi^a \ .$$

It is possible to choose transition functions for the gerbe to be holomorphic with respect to both complex structures. It is thus natural to talk about bi-holomorphic gerbes. Further details about the bi-holomorphic gerbe will be given in [22].

The term (3.21) is the pull-back of a locally exact form, which we denote b. Since the potential $K^{(\alpha)}$ is defined only locally over a patch U_α, this form is also defined only locally

$$b_\alpha = d(K_a^{(\alpha)} d\chi^a) , \qquad b_\alpha \in \Omega^2(U_\alpha) , \quad K^{(\alpha)} \in C^\infty(U_\alpha) \ . \tag{4.4}$$

Thus $\{b_\alpha\}$ is a collection of locally defined closed complex forms on M. These forms can be written as follows,

$$b_\alpha = (B_\alpha)_-^{(0,2)} - (B_\alpha)_+^{(2,0)} + \frac{i}{2}(\omega_- - \omega_+) \ . \tag{4.5}$$

The real and imaginary parts of b_α are

$$b_\alpha = \frac{i}{2}(F_\alpha^+ + F_\alpha^-) + \frac{1}{2}(B_\alpha)_- - \frac{1}{2}(B_\alpha)_+ \ = \frac{i}{2}(F_\alpha^+ + F_\alpha^-) + d\Lambda , \tag{4.6}$$

[3] Related discussions of gerbes in the context of WZW models may be found in, e.g., [20, 21].

where $F_\alpha^\pm \in \Omega^2(U_\alpha)$ are the real closed two-forms on U_α defined as follows

$$F_\alpha^\pm = i\left((B_\alpha)_\pm^{2,0} - (B_\alpha)_\pm^{0,2}\right) \mp \omega_\pm \ . \tag{4.7}$$

The property $dF_\alpha^\pm = 0$ is a consequence of the conditions (1.2) and (4.3). Thus, in a way the forms $F_\alpha^\pm$ are the local analogue of the Kähler form in the Kähler geometry.[4] The real part $d\Lambda$ is an exact form and does not contribute to the integral, so that the quasi-topological term in the action is

$$S_{\text{top}} = i\int\limits_{\Sigma_2} X^*(F) = i\int\limits_{X_*(\Sigma_2)} F\,, \tag{4.8}$$

where

$$F_\alpha = \frac{1}{2}(F_\alpha^+ + F_\alpha^-) \ . \tag{4.9}$$

Next, we check that this term reduces to the familiar topological terms in the standard A- and B-models. In the standard Kähler case when $J_- = -J_+$ and B is a globally defined closed two-form, F is the complexified Kähler class

$$iF = B + i\omega \ . \tag{4.10}$$

Then S_{top} is the topological term for the A-model as expected. In the standard Kähler case with $B = 0$ and $J_- = +J_+$, $F = 0$ and there is no topological term, as expected for the B-model. It is interesting that in this case, if we introduce a B which is a globally defined closed two-form, then

$$iF = -\left(B^{2,0} - B^{0,2}\right)\,,$$

so that

$$S_{\text{top}} = -\int \left(B^{2,0} - B^{0,2}\right)\,,$$

is the integral of a global 2-form and is well-defined despite the absence of an 'i'.

We now return to the general case in which B is a gerbe connection. By itself the term S_{top} is not well-defined. However we can make sense of S_{top} by exponentiating and interpreting this as a holonomy of a flat gerbe. Let us briefly recall the case of Wilson loops and flat connections on line bundles. For a line bundle with a flat connection, the connection A is a collection of locally defined closed 1-forms, with suitable transition functions. For a line bundle with connection A, the holonomy operator for a curve γ (a Wilson loop) is

$$W_A = \exp\left(i\oint\limits_\gamma A\right)\,, \tag{4.11}$$

[4] In Kähler geometry $\nabla J = 0$ is equivalent to $d\omega = 0$, while in our case $\nabla^\pm J_\pm = 0$ is locally equivalent to $dF_\pm = 0$.

and if A is flat this depends only on the homology class of γ. Then for flat line bundles, W_A defines a map

$$W_A \ : \ H_1(M) \ \rightarrow \ S^1 \, ,$$

so that the map is an element of $H^1(M, U(1))$.

The same idea works for a real flat gerbe. For a gerbe with a connection b there exists a holonomy operator defined for any 2-cycle Σ

$$W_b = \exp \left(i \oint_{\Sigma} b \right) \, , \tag{4.12}$$

and if the connection is flat, $db = 0$, this depends only on the homology class of Σ. Then for flat connections, W_b defines a map

$$W_b \ : \ H_2(M) \ \rightarrow \ S^1 \, ,$$

which is then an element of $H^2(M, U(1))$. The flat gerbe is a collection of locally defined closed 2-forms, with suitable transition functions. The operator W_b depends only on the homology of Σ and the connection b.

Now we would like to apply the idea of a flat gerbe to our quasi-topological term S_{top}. Upon exponentiating this term and interpreting as a holonomy operator

$$\exp \left(i \int_{X_*(\Sigma_2)} F \right) \ : \ H_2(M) \ \rightarrow \ S^1 \, , \tag{4.13}$$

we arrive at a term which depends only on the homology class of $X_*(\Sigma_2)$. Thus finally we conclude that, independent of the gauge, the topological term should be understood through the holonomy of a real flat gerbe connection,[5]

$$\exp \left(i \int_{\Sigma} F \right) = W_F = Hol \, (F) \, , \tag{4.14}$$

where $\Sigma = X_*(\Sigma_2)$.

5. Discussion

We have shown that the path integral can be written as a weighted sum for the B-twisted BiLP models, i.e., for twisted sigma models involving chiral and twisted chiral fields (and thus H-flux). One of the terms in the action corresponds to a flat gerbe connection and has an interpretation as a quasi-topological term in the quantized theory. Its exponential is a Wilson surface, a generalization of a Wilson line. We have further seen that the localization of the model is on holomorphic twisted chiral maps and constant chiral ones.

[5] For notation, see [10].

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A natural question is how to extend the discussion to the full Generalized Kähler Geometry, i.e., to include semi-chiral fields in the model. Indeed we believe that our result will extend to the general twisted $\mathcal{N} = (2, 2)$ model, including the semichiral fields. For the general case in which J_+ and J_- do not commute, then the $F^{\pm}$ defined by (4.7) still satisfy $dF^{\pm} = 0$ and so are flat gerbe connections. We conjecture that the twisted theory continues to be weighted by the holonomy of the gerbe connection $F = \frac{1}{2}(F^+ + F^-)$ in this general case, generalising the exponential of the degree. There are a number of important issues to be addressed regarding how to deal with the the $\mathcal{N} = (2, 2)$ Euclidean model. We plan to resolve these and related problems in the forthcoming publications.

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A. Gerbe holonomy

In this appendix we briefly review the notion of holonomy for line bundles and gerbes. For more details the reader may consult [10 − 13].

Consider a smooth manifold M with an open covering $\{U_\alpha\}$ where all open sets and intersections are contractible. The line bundle can be thought of as a set of transition functions

$$g_{\alpha\beta} \ : \ U_\alpha \cap U_\beta \ \rightarrow \ S^1 \, ,$$

which satisfy $g_{\alpha\beta} = g^{-1}_{\beta\alpha}$ and the cocycle condition on $U_\alpha \cap U_\beta \cap U_\gamma$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \ .$$

The connection on the line bundle can be defined as a collection of one-forms $A_\alpha \in \Omega^1(U_\alpha)$ such that on the double intersections $U_\alpha \cap U_\beta$

$$i A_\alpha - i A_\beta = g^{-1}_{\alpha\beta} d g_{\alpha\beta} \ .$$

Since on $U_\alpha \cap U_\beta$ $dA_\alpha = dA_\beta$ we can define a curvature two form ω on M such that $\omega = dA_\alpha$ on U_α. It can be shown that ω defines an integral cohomology class, $\omega/2\pi \in H^2(M, \mathbb{Z})$.

For any loop γ in M, the holonomy is defined as follows. First (assuming a suitably fine open cover) the loop γ is divided into segments γ_α such that each γ_α is in U_α and the point (if any) at which γ_α and γ_β join is denoted $\gamma_{\alpha\beta}$. Then the holonomy of A on the curve γ is

$$Hol(A, \gamma) = \exp\left(i \sum_{\gamma_\alpha} \int_{\gamma_\alpha} A_\alpha + i \sum_{\gamma_{\alpha\beta}} \log g_{\alpha\beta}(\gamma_{\alpha\beta}) \right) \, ,$$

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and it can be shown that it does not depend on a particular choice of the partition γ into $\{\gamma_\alpha, \gamma_{\alpha\beta}\}$. If $\omega = 0$, then there is flat connection on a line bundle. In this case the holonomy $Hol(A,\gamma)$ depends only on the homology class of γ.

A gerbe is a higher generalization of a line bundle. A gerbe can be defined as a set of transition functions on threefold intersections

$$g_{\alpha\beta\gamma} \ : \ U_\alpha \cap U_\beta \cap U_\gamma \ \to \ S^1 \,,$$

satisfying

$$g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\gamma\beta\alpha} \,, \qquad g_{\beta\gamma\delta} g_{\delta\gamma\alpha} g_{\alpha\beta\delta} g_{\beta\alpha\gamma} = 1 \,,$$

where the last condition is understood on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$. A connection on a gerbe is defined as a collection of one-forms and two-forms $\{A_{\alpha\beta}, B_\alpha\}$ such that $A_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta)$ and and $B_\alpha \in \Omega^2(U_\alpha)$ with the relations

$$iA_{\alpha\beta} + iA_{\beta\gamma} + iA_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma} \,,$$

on the triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ and

$$B_\alpha - B_\beta = dA_{\alpha\beta} \,,$$

on the double intersection $U_\alpha \cap U_\beta$. Since $dB_\alpha = dB_\beta$ on $U_\alpha \cap U_\beta$, one can define a curvature three form H on M such that $H = dB_\alpha$ on U_α. It can be shown that H defines an integral cohomology class, $H/2\pi \in H^3(M,\mathbb{Z})$.

For any closed 2-surface Σ in M, the holonomy of a gerbe with connection is defined as follows. We choose an open cover U_α of M and a simplicial decomposition of Σ into 2-simplices Σ_α such that Σ_α is in U_α and if Σ_α and Σ_β have a common edge, that 1-simplex is labelled $\Sigma_{\alpha\beta}$. If the three 1-simplices $\Sigma_{\alpha\beta}$, $\Sigma_{\beta\gamma}$, $\Sigma_{\gamma\beta}$ intersect in a point, it is labelled $\Sigma_{\alpha\beta\gamma}$. The gerbe holonomy is then

$$Hol(B,A,\Sigma) = \exp\left(i\sum_{\Sigma_\alpha} \int_{\Sigma_\alpha} B_\alpha + i\sum_{\Sigma_{\alpha\beta}} \int_{\Sigma_{\alpha\beta}} A_{\alpha\beta} + i\sum_{\Sigma_{\alpha\beta\gamma}} \log g_{\alpha\beta\gamma}(\Sigma_{\alpha\beta\gamma}) \right) \,,$$

and one can prove that this does not depend on the particular choice of open cover of M or of simplicial decomposition of Σ into $\{\Sigma_\alpha, \Sigma_{\alpha\beta}, \Sigma_{\alpha\beta\gamma}\}$. If $H = 0$ the gerbe is called flat. For a flat gerbe the holonomy depends only on the homology class of Σ.

References

[1] E. Witten, *Topological σ-models*, *Commun. Math. Phys.* **118** (1988) 411.

[2] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[3] J.M.F. Labastida and P.M. Llatas, *Topological matter in two-dimensions*, *Nucl. Phys.* **B 379** (1992) 220 [hep-th/9112051].

[4] K. Hori et al., *Mirror symmetry*, Clay Mathematics Monographs, American Mathematical Society (2003).

[5] J. Gates, S. J., C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear σ-models*, *Nucl. Phys.* **B 248** (1984) 157.

[6] P.S. Howe and G. Sierra, *Two-dimensional supersymmetric nonlinear σ-models with torsion*, *Phys. Lett.* **B 148** (1984) 451.

[7] M. Gualtieri, *Generalized complex geometry*, math.DG/0401221.

[8] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler manifolds and off-shell supersymmetry*, *Commun. Math. Phys.* **269** (2007) 833 [hep-th/0512164].

[9] T. Buscher, U. Lindström and M. Roček, *New supersymmetric σ-models with Wess-Zumino terms*, *Phys. Lett.* **B 202** (1988) 94.

[10] D.M. Belov, C.M. Hull and R. Minasian, *T-duality, gerbes and loop spaces*, arXiv:0710.5151.

[11] N. Hitchin, *Lectures on special Lagrangian submanifolds*, in *Winter School on Mirror symmetry, Vector Bundles and Lagrangian Submanifolds*, C. Vafa and S.-T. Yau (eds.), Studies in Advanced Mathematics 23, AMS/International Press, Providence (2001) pgg. 151-182.

[12] D.S. Chatterjee, *On gerbs*, PhD thesis at the University of Cambridge (1998).

[13] M. Mackaay and R. Picken, *The holonomy of gerbes with connections*, math.DG/0007053.

[14] A. Kapustin and Y. Li, *Topological σ-models with H-flux and twisted generalized complex manifolds*, hep-th/0407249.

[15] A. Bredthauer, U. Lindström, J. Persson and M. Zabzine, *Generalized Kähler geometry from supersymmetric σ-models*, *Lett. Math. Phys.* **77** (2006) 291 [hep-th/0603130].

[16] R. Zucchini, *The biHermitian topological σ-model*, *JHEP* **12** (2006) 039 [hep-th/0608145].

[17] W.-y. Chuang, *Topological twisted σ-model with H-flux revisited*, *J. Phys.* **A 41** (2008) 115402 [hep-th/0608119].

[18] Imperial-Stony Brook-Uppsala collaboration in progress.

[19] C.M. Hull, U. Lindström, L. Melo dos Santos, R. von Unge and M. Zabzine, *Euclidean supersymmetry, twisting and topological σ-models*, *JHEP* **06** (2008) 031 [arXiv:0805.3321].

[20] K. Gawędzki and N. Reis, *WZW branes and gerbes*, *Rev. Math. Phys.* **14** (2002) 1281 [hep-th/0205233].

[21] U. Schreiber, C. Schweigert and K. Waldorf, *Unoriented WZW models and holonomy of bundle gerbes*, *Commun. Math. Phys.* **274** (2007) 31 [hep-th/0512283].

[22] C. Hull, U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler geometry and gerbes*, to appear.

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